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Spin and Statistics on the Groenewold-Moyal Plane:Pauli-Forbidden Levels and Transitions

A.P. Balachandran[a*], G. Mangano[a,b†], A. Pinzul[a‡] and S. Vaidya[c§]

[a] Department of Physics, Syracuse University, Syracuse NY, 13244-1130, USA.

[b] INFN, Sezione di Napoli and Dipartimento di Scienze Fisiche, Università di Napoli *Federico II*, Via Cintia, I-80126 Napoli, Italy.

[c] Centre for High Energy Physics, Indian Institute of Science, Bangalore, 560012, India.

Abstract

The Groenewold-Moyal plane is the algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ of functions on $\mathbb{R}^{d+1}$ with the $*$-product as the multiplication law, and the commutator $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$ $(\mu, \nu = 0, 1, ..., d)$ between the coordinate functions. Chaichian et al. [1] and Aschieri et al. [2] have proved that the Poincaré group acts as automorphisms on $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ if the coproduct is deformed. (See also the prior work of Majid [3], Oeckl [4] and Grosse et al [5]). In fact, the diffeomorphism group with a deformed coproduct also does so according to the results of [2]. In this paper we show that for this new action, the Bose and Fermi commutation relations are deformed as well. Their potential applications to the quantum Hall effect are pointed out. Very striking consequences of these deformations are the occurrence of Pauli-forbidden energy levels and transitions. Such new effects are discussed in simple cases.

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Dedicated to Rafael Sorkin, our friend and teacher, and a true and creative seeker of knowledge.

*bal@phy.syr.edu

†mangano@na.infn.it

‡apinzul@phy.syr.edu

§vaidya@cts.iisc.ernet.in

1 Introduction

The Groenewold-Moyal plane is the algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ of functions on $\mathbb{R}^{d+1}$ with the $*$-product $\alpha *_\theta \beta$ between functions α and β as the product law, where

$$\begin{aligned} \alpha *_\theta \beta\ (x) &= \left[\alpha \exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right)\beta\right](x)\ , \\ \theta^{\mu\nu} &= -\theta^{\nu\mu} \in \mathbb{R}\ ,\ x = (x^0, x^1, \ldots, x^d)\ . \end{aligned} \tag{1.1}$$

The Poincaré group $\mathcal{P}$ acts on $\mathbb{R}^{d+1}$ and hence on its smooth functions $C^\infty(\mathbb{R}^{d+1})$ regarded just as a vector space. If $g \in \mathcal{P}$ and $g : x \to gx$, then for $\gamma \in C^\infty(\mathbb{R}^{d+1})$

$$(g\gamma)(x) = \gamma(g^{-1}x)\ . \tag{1.2}$$

However, in general

$$(g\alpha) *_\theta (g\beta) \neq g(\alpha *_\theta \beta)\ , \tag{1.3}$$

so that this action of $\mathcal{P}$ is not an automorphism of $\mathcal{A}_\theta(\mathbb{R}^{d+1})$.

Similar remarks can be made generically about any group which acts on $\mathbb{R}^{d+1}$ and in particular about the diffeomorphism group $\mathcal{D}$. Only a limited group of transformations, such as translations, gives the equality in (1.3). Nevertheless, there is a way to avoid this limitation with $\mathcal{D}$. It involves introducing a new deformed coproduct Δ_θ on $\mathcal{D}$. The revival of this idea in recent times is due to [1, 2, 4, 6]. But its origins can be traced back to Drin'feld [7] in mathematics. This Drin'feld twist leads naturally to deformed R-matrices and statistics for quantum groups, as discussed by Majid [3]. Subsequently, Fiore and Schupp [8] and Watts [9, 10] explored the significance of the Drin'feld twist and R-matrices while Fiore [11, 12] and Fiore and Schupp [13], Oeckl [4] and Grosse et al [5] studied the importance of R-matrices for statistics. Oeckl [4] and Grosse et al [5] also developed quantum field theories using different and apparently inequivalent approaches, the first on the Moyal plane and the second on the q-deformed fuzzy sphere. Recent work, including ours, has significant overlap with the earlier literature. We share many features in particular with [4, 5].

In [2,6] the authors focused on $\mathcal{D}$ and developed Riemannian geometry and gravity theories based on Δ_θ, while [1] focused on the Poincaré subgroup $\mathcal{P}$ of $\mathcal{D}$ and explored the consequences of Δ_θ for quantum field theories. Twisted conformal symmetry was discussed by [14]. We explain the basics of all this work in Section 2.

In Section 3, we discuss the impact of the deformed tensor product on the Bose and Fermi commutation relations. In fact, they are also deformed. We give an explicit formula for the new creation-annihilation operators in terms of the standard ($\theta^{\mu\nu} = 0$) ones. State vectors can still be classified by the irreducible representations of the permutation group, but the action of the latter on the Hilbert space is deformed as well.

Previous research on the spin-statistics theorem on $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ is due to Alvarez-Gaumé and Vazquez-Mozo [15], but they do not use the deformed coproduct on $\mathcal{P}$.

In Section 4, we construct the second quantization formalism corresponding to the deformed commutation relations, introducing also the corresponding symmetry under permutations of physical states.

In Section 5, we argue that excitations in the quantum Hall effect should be described by deformed statistics.

Finally, in Section 6, we discuss the possible phenomenological implications of the deformed commutation relations, considering in particular the case of systems of fermionic identical particles. We show that there exist state vectors of the system which violate the Pauli exclusion principle. There are quite stringent tests on Pauli violating transitions in nuclear (see for example [16, 17] and references therein) and atomic systems [18], and crystals [19], so that the energy scale associated with $\theta^{\mu\nu}$ (whose dimension is inverse squared energy) can be severely constrained. This issue will be studied in more detail later.

In another work [20], it is proved that UV-IR mixing is entirely absent for quantum field theories on $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ with the deformed statistics.

2 The Deformed Coproduct

2.1 *Tensor Product of Representations*

Suppose that a group G acts on a complex vector space V by a representation ρ. We denote this action by

$$v \to \rho(g)v \ , \tag{2.1}$$

for $g \in G$ and $v \in V$. Then the group algebra G^* also acts on V. A typical element of G^* is

$$\int dg\, \alpha(g)\, g, \quad \alpha(g) \in \mathbb{C} \ , \tag{2.2}$$

where dg is a measure on G. Its action is

$$v \to \int dg\, \alpha(g)\, \rho(g)\, v \ . \tag{2.3}$$

Both G and G^* act on $V \otimes_{\mathbb{C}} V$, the tensor product of V's over $\mathbb{C}$, as well. These actions are usually taken to be

$$v_1 \otimes v_2 \to [\rho(g) \otimes \rho(g)]\,(v_1 \otimes v_2) = \rho(g)v_1 \otimes \rho(g)v_2 \ , \tag{2.4}$$

and

$$v_1 \otimes v_2 \to \int dg\, \alpha(g)\, \rho(g)v_1 \otimes \rho(g)v_2 \tag{2.5}$$

respectively, for $v_1, v_2 \in V$.

In Hopf algebra theory, the action of G and G^* on tensor products is formalized using the coproduct Δ, a homomorphism from G^* to $G^* \otimes G^*$, which on restriction

to G gives a homomorphism from G to $G^* \otimes G^*$. This restriction specifies Δ on all of G^* by linearity. Thus if

$$\Delta:\ g \to \Delta(g)\ , \tag{2.6}$$

$$\Delta(g_1)\Delta(g_2) = \Delta(g_1 g_2)\ , \tag{2.7}$$

we have

$$\Delta\left(\int dg\, \alpha(g)\, g\right) = \int dg\, \alpha(g)\, \Delta(g)\ . \tag{2.8}$$

For the familiar choice $\Delta(g) = g \otimes g$, the action (2.4) can be written as

$$v_1 \otimes v_2 \to [\rho \otimes \rho]\, \Delta(g) v_1 \otimes v_2\ . \tag{2.9}$$

But any choice of coproduct will do to define an action of G on $V \otimes V$ using (2.9).

Likewise, if G acts on vector spaces V and W by representations ρ and σ, respectively, and Δ is a coproduct on G, G can act on $V \otimes W$ according to

$$v \otimes w \to [\rho \otimes \sigma]\, \Delta(g) v \otimes w\ , \tag{2.10}$$

for $v \in V$, $w \in W$. This action extends by linearity to an action of G^*.

Not all choices of Δ are equivalent. In particular the irreducible representations (IRR's), which can occur in the reduction of $[\rho \otimes \sigma]$ can depend upon Δ. Examples of this sort perhaps occur for the Poincaré group.

2.2 *The Carrier of Group Action is an Algebra*

Until now we assumed only that V, W are vector spaces. Suppose next that V is an algebra $\mathcal{A}$ (over $\mathbb{C}$). In that case, as discussed by [1, 2] there is also a compatibility condition on Δ. It comes about as follows.

As $\mathcal{A}$ is an algebra, we have a rule for taking products of elements of $\mathcal{A}$. That means that there is a multiplication map

$$\begin{aligned} m : \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A}\ , \\ \alpha \otimes \beta &\to m(\alpha \otimes \beta)\ , \end{aligned} \tag{2.11}$$

for $\alpha, \beta \in \mathcal{A}$, the product $\alpha\beta$ being $m(\alpha \otimes \beta)$.

It is now essential that Δ be compatible with m. That means that if we transform $\alpha \otimes \beta$ by g-action and then apply m, it should be equal to the g-transform of $m(\alpha \otimes \beta)$:

$$m\left((\rho \otimes \rho)\Delta(g)\, (\alpha \otimes \beta)\right) = \rho(g) m(\alpha \otimes \beta)\ . \tag{2.12}$$

This result is encoded in the commutative diagram

$$\begin{array}{ccc} \alpha \otimes \beta & \longrightarrow & (\rho \otimes \rho)\Delta(g)\alpha \otimes \beta \\ m\ \downarrow & & \downarrow\ m \\ m(\alpha \otimes \beta) & \longrightarrow & \rho(g) m(\alpha \otimes \beta) \end{array} \tag{2.13}$$

If such a Δ can be found, G is an automorphism of $\mathcal{A}$. In the absence of such a Δ, G does not act on $\mathcal{A}$.

2.3 *The Case of the Groenewold-Moyal Plane*

In the Groenewold-Moyal plane, the multiplication map depends on $\theta^{\mu\nu}$ and will be denoted by m_θ. It is defined by

$$m_\theta(\alpha \otimes \beta) = m_0\left(e^{-\frac{i}{2}(-i\partial_\mu)\theta^{\mu\nu}\otimes(-i\partial_\nu)}\alpha \otimes \beta\right) \ , \tag{2.14}$$

where m_0 is the point-wise multiplication

$$m_0(\gamma \otimes \delta)(x) := \gamma(x)\delta(x) \tag{2.15}$$

of any two functions γ and δ.

We introduce the notation

$$F_\theta = e^{-\frac{i}{2}(-i\partial_\mu)\theta^{\mu\nu}\otimes(-i\partial_\nu)} \tag{2.16}$$

for the factor appearing in (2.14) so that

$$m_\theta(\alpha \otimes \beta) = m_0\left(F_\theta \alpha \otimes \beta\right) \ . \tag{2.17}$$

Let $g \in \mathcal{D}$ act on $\mathbb{R}^{d+1}$ by $x \to gx$ and hence on functions by $\alpha \to \rho(g)\alpha$ where the representation ρ is canonical:

$$(\rho(g)\alpha)(x) = \alpha(g^{-1}x) \ . \tag{2.18}$$

(This action was denoted in Eq.(1.2) omitting the symbol ρ.) The important observation is that it can act on $\mathcal{A}_\theta(\mathbb{R}^{d+1}) \otimes \mathcal{A}_\theta(\mathbb{R}^{d+1})$ as well compatibly with m_θ if a new coproduct Δ_θ is used, where

$$\Delta_\theta(g) = e^{\frac{i}{2}P_\mu\otimes\theta^{\mu\nu}P_\nu}(g\otimes g)e^{-\frac{i}{2}P_\mu\otimes\theta^{\mu\nu}P_\nu} = \hat{F}_\theta^{-1}(g \otimes g)\hat{F}_\theta \ , \tag{2.19}$$

P_μ being the generators of translations. On functions, that is, in the representation ρ, it becomes $-i\partial_\mu$, so that the two factors in (2.19) can be expressed in terms of F_θ and its inverse.

We can check that Δ_θ is compatible with m_θ as follows

$$\begin{aligned} m_\theta\left((\rho \otimes \rho)\Delta_\theta(g)(\alpha \otimes \beta)\right) &= m_0\left(F_\theta(F_\theta^{-1}\rho(g) \otimes \rho(g)F_\theta)\alpha \otimes \beta\right) \\ &= \rho(g)\left(\alpha *_\theta \beta\right), \quad \alpha, \beta \in \mathcal{A}_\theta(\mathbb{R}^{d+1}) \end{aligned} \tag{2.20}$$

as required.

The action of the Poincaré group on tensor products of plane waves is simple. For the momentum $p = (p_0, p_1, ...p_d) \in \mathbb{R}^{d+1}$, let $\mathbf{e}_p \in \mathcal{A}_\theta(\mathbb{R}^{d+1})$ where

$$\mathbf{e}_p(x) = e^{ip\cdot x}, \quad p\cdot x = p_\mu x^\mu \ . \tag{2.21}$$

In the case of the Poincaré group, if $\exp(iP{\cdot}a)$ is a translation,

$$(\rho \otimes \rho)\Delta_\theta \left(e^{iP\cdot a}\right) \mathbf{e}_p \otimes \mathbf{e}_q = e^{i(p+q)\cdot a} \mathbf{e}_p \otimes \mathbf{e}_q \ , \tag{2.22}$$

while if Λ is a Lorentz transformation

$$(\rho \otimes \rho)\Delta_\theta(\Lambda)\mathbf{e}_p \otimes \mathbf{e}_q = \left[e^{\frac{i}{2}(\Lambda p)_\mu \theta^{\mu\nu} (\Lambda q)_\nu} e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu}\right] \mathbf{e}_{\Lambda p} \otimes \mathbf{e}_{\Lambda q} \ . \tag{2.23}$$

Thus the coproduct on translations is not affected while the coproduct for the Lorentz group is changed.

2.4 *Action on Fourier Coefficients*

If φ is a scalar field, we can regard it as an element of $\mathcal{A}_\theta(\mathbb{R}^{d+1})$. If its Fourier representation is

$$\varphi = \int d\mu(p)\, \tilde{\varphi}(p)\, \mathbf{e}_p \ , \tag{2.24}$$

where $d\mu(p)$ is a Lorentz-invariant measure, then

$$\rho(\Lambda)\varphi = \int d\mu(p)\, \tilde{\varphi}(p)\, \mathbf{e}_{\Lambda p} = \int d\mu(p)\, \tilde{\varphi}(\Lambda^{-1} p)\, \mathbf{e}_p \ , \tag{2.25}$$

$$\rho\left(e^{iP\cdot a}\right)\varphi = \int d\mu(p)\, e^{ip\cdot a} \tilde{\varphi}(p)\, \mathbf{e}_p \ . \tag{2.26}$$

Thus the representation $\tilde{\rho}$ of the Poincaré group on $\tilde{\varphi}$ is specified by

$$(\tilde{\rho}(\Lambda)\tilde{\varphi})(p) = \tilde{\varphi}(\Lambda^{-1} p) \ , \tag{2.27}$$

$$\left(\tilde{\rho}\left(e^{iP\cdot a}\right)\tilde{\varphi}\right)(p) = e^{ip\cdot a}\tilde{\varphi}(p) \ . \tag{2.28}$$

If χ is another field of $\mathcal{A}_\theta(\mathbb{R}^{d+1})$,

$$\chi = \int d\mu(p)\, \tilde{\chi}(p)\, \mathbf{e}_p \ , \tag{2.29}$$

then

$$\varphi \otimes \chi = \int d\mu(p)\, d\mu(q)\, \tilde{\varphi}(p)\tilde{\chi}(q)\, \mathbf{e}_p \otimes \mathbf{e}_q \ . \tag{2.30}$$

Using (2.22), we see that the action of translations on $\tilde{\varphi} \otimes \tilde{\chi}$ is

$$\Delta_\theta\left(e^{iP\cdot a}\right)(\tilde{\varphi} \otimes \tilde{\chi})(p, q) = e^{i(p+q)\cdot a} \tilde{\varphi}(p)\tilde{\chi}(q) \ . \tag{2.31}$$

Using (2.31) we can deduce the action of twisted Lorentz transformations to be

$$\Delta_\theta(\Lambda)(\tilde{\varphi} \otimes \tilde{\chi})(p, q) = \tilde{F}_\theta^{-1}\left(\Lambda^{-1} p, \Lambda^{-1} q\right) \tilde{F}_\theta(p, q)\, \tilde{\varphi}(\Lambda^{-1} p)\tilde{\chi}(\Lambda^{-1} q) \ . \tag{2.32}$$

Here

$$\tilde{F}_\theta\,(r,s) := e^{-\frac{i}{2} r_\mu \theta^{\mu\nu} s_\nu} \tag{2.33}$$

and we have omitted writing $\rho \otimes \rho$ in front of Δ_θ's.

We remark that had we used (2.23) to deduce the transformation law for the Fourier coefficients, we would have got $\tilde{F}_\theta\,(\Lambda^{-1}p, \Lambda^{-1}q)\,\tilde{F}_\theta^{-1}\,(p,q)\,\tilde{\varphi}(\Lambda^{-1}p)\tilde{\chi}(\Lambda^{-1}q)$ for the right-hand side of (2.32). We will use (2.32) hereafter as it can be deduced from the conventional action of P_μ given by (2.31).

3 Quantum Fields and Spin-Statistics

A free relativistic scalar quantum field φ of mass m can be expanded as

$$\varphi = \int \frac{d^d p}{2p_0} \left(a(p)\,\mathbf{e}_p + a^\dagger(p)\mathbf{e}_{-p} \right) \ , \tag{3.1}$$

where $p_0 = \sqrt{|\vec{p}|^2 + m^2} \geq 0$, and $a(p), a^\dagger(p)$ are subject to suitable relations to be stated below. If $c(p), c^\dagger(p)$ are the limits of these operators when $\theta^{\mu\nu} = 0$, these relations are

$$\begin{aligned} [c(p), c(q)] &= \left[c^\dagger(p), c^\dagger(q)\right] = 0 \ , &(3.2)\\ \left[c(p), c^\dagger(q)\right] &= 2p_0 \delta^d(p-q) \ . &(3.3) \end{aligned}$$

We now argue that such relations are incompatible for $\theta^{\mu\nu} \neq 0$. Rather $a(p)$ and $a^\dagger(p)$ fulfill certain deformed relations which reduce to (3.2), (3.3) for $\theta^{\mu\nu} = 0$. We may therefore say that statistics is deformed, though this is not entirely precise, as we discuss later.

Similar deformations occur for the operator relations of all tensorial and spinorial quantum fields.

Suppose now that

$$a(p)a(q) = \tilde{G}_\theta(p,q)a(q)a(p) \ , \tag{3.4}$$

where $\tilde{G}_\theta$ is a $\mathbb{C}$-valued function of p and q yet to be determined. In particular, if $U(\Lambda)$ and $U(\exp(iP{\cdot}a))$ are the operators implementing the Lorentz transformations and translations respectively on the quantum Hilbert space,

$$\begin{aligned} U(\Lambda)\tilde{G}_\theta(p,q)U(\Lambda)^{-1} &= \tilde{G}_\theta(p,q) \ , &(3.5)\\ U(\exp(iP{\cdot}a))\tilde{G}_\theta(p,q)U(\exp(iP{\cdot}a))^{-1} &= \tilde{G}_\theta(p,q) \ . &(3.6) \end{aligned}$$

The transformations of $a(p)a(q) = (a \otimes a)(p,q)$ and $a(q)a(p)$ are determined by Δ_θ. Hence conjugating (3.4) by $U(\Lambda)$, we get

$$\begin{aligned} &\tilde{F}_\theta^{-1}\left(\Lambda^{-1}p, \Lambda^{-1}q\right)\tilde{F}_\theta\,(p,q)\,a(\Lambda^{-1}p)a(\Lambda^{-1}q) = \\ &= \tilde{G}_\theta(p,q)\tilde{F}_\theta^{-1}\left(\Lambda^{-1}q, \Lambda^{-1}p\right)\tilde{F}_\theta\,(q,p)\,a(\Lambda^{-1}q)a(\Lambda^{-1}p) \ , \end{aligned} \tag{3.7}$$

or, on using $\tilde{F}_\theta(r,s) = \tilde{F}_\theta^{-1}(s,r)$,

$$a(\Lambda^{-1}p)a(\Lambda^{-1}q) = \tilde{G}_\theta(p,q)\tilde{F}_\theta^{-2}\left(\Lambda^{-1}q, \Lambda^{-1}p\right)\tilde{F}_\theta^{2}(q,p)\, a(\Lambda^{-1}q)a(\Lambda^{-1}p)\ . \qquad (3.8)$$

Using (3.4) after changing p to $\Lambda^{-1}p$ and q to $\Lambda^{-1}q$, we get

$$\tilde{G}_\theta(\Lambda^{-1}p, \Lambda^{-1}q)\tilde{F}_\theta^{2}\left(\Lambda^{-1}q, \Lambda^{-1}p\right) = \tilde{G}_\theta(p,q)\tilde{F}_\theta^{2}(q,p)\ , \qquad (3.9)$$

whose solution is

$$\tilde{G}_\theta(p,q) = \tilde{\eta}(p,q)\tilde{F}_\theta^{-2}(q,p)\ , \qquad (3.10)$$

where $\tilde{\eta}$ is a Lorentz-invariant function of p and q. For $\theta^{\mu\nu} = 0$, φ is a standard scalar field and $\tilde{\eta}(p,q)$ takes the constant value $\eta = +1$. So it is natural to take

$$\tilde{\eta}(p,q) = \eta = +1, \quad \text{for all } \theta^{\mu\nu}\ , \qquad (3.11)$$

even though $\tilde{\eta}(p,q)$ can depend on p, q and $\theta^{\mu\nu}$ and approach the value $+1$ only in the limit of vanishing $\theta^{\mu\nu}$.

Note that in $1+1$ dimensions, $\tilde{F}_\theta(\Lambda p, \Lambda q) = \tilde{F}_\theta(p,q)$ is itself Lorentz-invariant (but not invariant under parity). Also, $2+1$ dimensions is special because of the availability of braid statistics. Thus for anyons, $\tilde{\eta}(p,q)$ can be taken to be a fixed phase.

Summarizing

$$a(p)a(q) = \eta\tilde{F}_\theta^{-2}(q,p)\, a(q)a(p)\ . \qquad (3.12)$$

The creation operator $a^\dagger(q)$ carries momentum $-q$, hence its deformed relation for scalar fields is

$$a(p)a^\dagger(q) = \tilde{\eta}'(p,q)\tilde{F}_\theta^{-2}(-q,p)\, a^\dagger(q)a(p) + 2p_0\delta^d(p-q)\ . \qquad (3.13)$$

There is no need that $\tilde{\eta}(p,q) = \tilde{\eta}'(p,q)$, even though as $\theta^{\mu\nu}$ approaches zero we require that $\tilde{\eta}'(p,q)$ approaches the constant $\eta' = +1$. Hence, as before we will set $\tilde{\eta}'(p,q) = \eta' = +1$ for all $\theta^{\mu\nu}$.

Finally, the adjoint of (3.12) gives

$$\bar{\eta}a^\dagger(p)a^\dagger(q) = \tilde{F}_\theta^{-2}(q,p)\, a^\dagger(q)a^\dagger(p)\ , \qquad (3.14)$$

where $\bar{\eta} = +1$ for $\eta = +1$.

For spinorial free fields, there are similar deformed relations where the factors $\tilde{\eta}, \tilde{\eta}'$ approach -1 as $\theta^{\mu\nu} \to 0$.

4 Construction of Deformed Oscillators from Undeformed Oscillators

We have presented such a construction elsewhere [21] when considering deformations of target manifolds of fields.

Let $c(p), c^\dagger(p)$ denote the undeformed oscillators, the limits of $a(p), a^\dagger(p)$ when $\theta^{\mu\nu} \to 0$, as in (3.2), (3.3). Then

$$a(p) = c(p) e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} \ , \tag{4.1}$$

where P_ν generates translations in the Hilbert space:

$$P_\nu = \int \frac{d^d p}{2p_0} \ p_\nu c^\dagger(p) c(p) \ , \tag{4.2}$$

$$[P_\nu, a(p)] = -p_\nu a(p), \quad [P_\nu, a^\dagger(p)] = p_\nu a^\dagger(p) \ . \tag{4.3}$$

The adjoint of (4.1) also gives

$$a^\dagger(p) = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c^\dagger(p) \ . \tag{4.4}$$

Before checking that this ansatz for the a-oscillators works, let us first point out that

$$c(p) e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} = e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c(p) \ , \tag{4.5}$$

$$e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c^\dagger(p) = c^\dagger(p) e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} \ , \tag{4.6}$$

due to the antisymmetry of $\theta^{\mu\nu}$. Hence the ordering of factors in (4.1) is immaterial. Note also that

$$P_\nu = \int \frac{d^d p}{2p_0} p_\nu a^\dagger(p) a(p) \ , \tag{4.7}$$

so that the map from the c- to the a-oscillators is invertible.

We can check the relation (3.12) as follows. We have

$$c(p) e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c(q) e^{\frac{i}{2} q_\rho \theta^{\rho\sigma} P_\sigma} = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu} c(p) c(q) e^{\frac{i}{2} (p+q)_\mu \theta^{\mu\nu} P_\nu} \ . \tag{4.8}$$

Hence since $[c(p), c(q)] = 0$ and $\theta^{\mu\nu} = -\theta^{\nu\mu}$, we get (3.12) with $\eta = +1$ for $a(p)$ defined by (4.1). We can check the remaining commutation relations as well in the same way.

4.1 *Deformed Permutation Symmetry*

Let $\mathcal{F}$ be the Fock space of the c-oscillators. Since the a-oscillators can be constructed from the c's, $\mathcal{F}$ is also a representation space for the a-oscillators. In particular, the Fock vacuum is annihilated by $a(p)$:

$$a(p)|0\rangle = 0 \ . \tag{4.9}$$

We work with the representation of the a-oscillators on $\mathcal{F}$.

Multi-particle vector states for $\theta^{\mu\nu} \neq 0$ are obtained by applying polynomials of $a^\dagger(p)$'s on $|0\rangle$.

The number operator

$$N = \int \frac{d^d p}{2p_0} c^\dagger(p) c(p) \ , \tag{4.10}$$

has the same expression in terms of $a(p)$'s and $a^\dagger(p)$'s,

$$N = \int \frac{d^d p}{2p_0} a^\dagger(p) a(p) \ , \tag{4.11}$$

and has the standard commutators with these oscillators,

$$[N, a^\dagger(p)] = a^\dagger(p), \quad [N, a(p)] = -a(p) \ . \tag{4.12}$$

Thus

$$N \prod_{i=1}^{k} a^\dagger(p_i)^{n_i} |0\rangle = \left(\sum_{j=1}^{k} n_j \right) \left(\prod_{i=1}^{k} a^\dagger(p_i)^{n_i} \right) |0\rangle \ , \tag{4.13}$$

and we can justifiably call

$$\prod_{i=1}^{k} (a^\dagger(p_i))^{n_i} |0\rangle \ , \tag{4.14}$$

as the n-particle state where $n = \sum_{i=1}^{k} n_i$.

We now show that there is a totally symmetric representation of the permutation group on these vectors. The operator representatives of its group elements depend on $\theta^{\mu\nu}$, but they reduce to the standard realizations for $\theta^{\mu\nu} = 0$.

First consider the free tensor product of two single particle states. On these, we can define the transposition $\hat{\sigma}$,

$$\hat{\sigma}(v(p) \otimes v(q)) := v(q) \otimes v(p) \ , \tag{4.15}$$

where

$$v(p) = a^\dagger(p)|0\rangle \ , \tag{4.16}$$

and so

$$\hat{\sigma}^2 = 1\!\!1 \ . \tag{4.17}$$

Here there is no relation between $v(p) \otimes v(q)$ and $v(q) \otimes v(p)$ for a generic v and all p, q.

The twist

$$\hat{F}_\theta = e^{-\frac{i}{2} P_\mu \theta^{\mu\nu} \otimes P_\nu} \tag{4.18}$$

acts on $v(p) \otimes v(q)$ as

$$\hat{F}_\theta (v(p) \otimes v(q)) = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu} v(p) \otimes v(q) \ . \tag{4.19}$$

By the antisymmetry of $\theta^{\mu\nu}$,

$$\hat{F}_\theta \hat{\sigma} = \hat{\sigma} \hat{F}_\theta^{-1} \ , \tag{4.20}$$

so that

$$\hat{T} = \hat{F}_\theta^{-2} \hat{\sigma} \ , \tag{4.21}$$

is an involution:

$$\hat{T}^2 = \mathbb{1} \ . \tag{4.22}$$

Note that the action of neither $\hat{\sigma}$ nor $\hat{F}_\theta^{-2}$ preserves the relation (3.14), while that of $\hat{T}$ does for $\bar{\eta} = +1$. That is, if (3.14) is true with $\bar{\eta} = +1$, then so is

$$\hat{T} a^\dagger(p) a^\dagger(q) \hat{T}^{-1} = \tilde{F}_\theta^{-2}(q,p) \hat{T} a^\dagger(q) a^\dagger(p) \hat{T}^{-1} \ . \tag{4.23}$$

This means that $\hat{F}_\theta^{-2}$ and $\hat{\sigma}$ individually map the subspace $\mathcal{H}_S$ spanned by the vectors $\{a^\dagger(p)a^\dagger(q)|0\rangle\}$ out of $\mathcal{H}_S$ and into the full free tensor product of two single particle subspaces, while $\hat{F}_\theta^{-2}\hat{\sigma}$ maps $\mathcal{H}_S$ to $\mathcal{H}_S$.

Further by (3.14),

$$\hat{T} a^\dagger(p) a^\dagger(q)|0\rangle = a^\dagger(p) a^\dagger(q)|0\rangle \ . \tag{4.24}$$

For $\theta^{\mu\nu} = 0$, we recover $\hat{T} = \hat{\sigma}$, the standard representation. We therefore call $a^\dagger(p)a^\dagger(q)|0\rangle$ as the symmetric state. Bose symmetry thus generalizes to symmetry under $\hat{T}$.

The generalizations of $\hat{T}$ to three-particle states are the two transpositions

$$\hat{T}_{12} = \hat{T} \otimes \mathbb{1}, \quad \hat{T}_{23} = \mathbb{1} \otimes \hat{T} \ . \tag{4.25}$$

On the n-particle states, $\hat{T}$ generalizes to the $(n-1)$ transpositions

$$\hat{T}_{i,i+1} = \underbrace{\mathbb{1} \otimes ... \otimes}_{(i-1) \ \text{factors}} \hat{T} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1} ... \otimes \mathbb{1}}_{n-(i+1) \ \text{factors}} \ . \tag{4.26}$$

They square to unity:

$$\hat{T}_{i,i+1}^2 = \mathbb{1} \ . \tag{4.27}$$

In addition, as one can easily verify, they fulfill the relation

$$\hat{T}_{i,i+1} \hat{T}_{i+1,i+2} \hat{T}_{i,i+1} = \hat{T}_{i+1,i+2} \hat{T}_{i,i+1} \hat{T}_{i+1,i+2} \ . \tag{4.28}$$

In view of (4.27) and (4.28) and a known theorem [22], $\hat{T}_{i,i+1}$ generate the permutation group S_n.

The preceding discussion shows that we get the totally symmetric representation of S_n on the physical state vectors of a scalar field: the scalar field describes generalized bosons.

4.2 *Projector for Physical States*

Let $\hat{t}_i,\ (i = 1, ..., n!)$ be the representatives of the elements of S_n on $\mathcal{F}$. The $\hat{t}_i$ can be written in terms of $\hat{T}_{i,i+1}$. Then, as is well-known [23],

$$\hat{\mathcal{P}} = \frac{1}{n!}\left(\sum_i \hat{t}_i\right) \ , \tag{4.29}$$

is the projector to the symmetric representations of S_n carried by $\mathcal{F}$. The physical space is

$$\hat{\mathcal{P}}\mathcal{F} \ . \tag{4.30}$$

4.3 *Observables*

Observables $\hat{K}$ must preserve the space $\hat{\mathcal{P}}\mathcal{F}$:

$$\hat{K}\hat{\mathcal{P}}\mathcal{F} \subseteq \hat{\mathcal{P}}\mathcal{F} \ . \tag{4.31}$$

Hence they must commute with $\hat{\mathcal{P}}$,

$$\hat{K}\hat{\mathcal{P}} = \hat{\mathcal{P}}\hat{K} \ . \tag{4.32}$$

This is assured if they commute with the permutations:

$$\hat{T}_{i,i+1}\hat{\mathcal{P}} = \hat{\mathcal{P}}\hat{T}_{i,i+1} \ , \tag{4.33}$$

$$\hat{t}_i\hat{\mathcal{P}} = \hat{\mathcal{P}}\hat{t}_i \ . \tag{4.34}$$

Let us check that the Poincaré transformations commute with permutations. (They will, of course, since we arrived at the deformed representation of permutations by requiring Poincaré invariance.) If U is the representation of the Poincaré group with elements g on the one-particle quantum states, then its representation in say two-particle states is $(U \otimes U)\Delta_\theta$. The image of g in this representation is hence

$$U^{(2)}(g) := \hat{F}_\theta^{-1}[U(g) \otimes U(g)]\hat{F}_\theta \ . \tag{4.35}$$

Now

$$\begin{aligned} \hat{T}U^{(2)}(g) &= \hat{F}_\theta^{-1}\hat{\sigma}[(U(g) \otimes U(g)]\hat{F}_\theta = \hat{F}_\theta^{-1}[U(g) \otimes U(g)]\hat{\sigma}\hat{F}_\theta \\ &= \hat{F}_\theta^{-1}[U(g) \otimes U(g)]\hat{F}_\theta^{-1}\hat{\sigma} = U^{(2)}(g)\hat{T} \ . \end{aligned} \tag{4.36}$$

This proof generalizes to the n-particle sectors. Hence Poincaré transformations commute with permutations.

5 Quantum Hall System

Consider a particle of charge e, like the electron, moving in the $1-2$-plane $\mathbb{R}^2 \subset \mathbb{R}^3$ in a constant magnetic field B directed in the third direction. The quantum mechanical degrees of freedom can be described by two sets of mutually commuting oscillators $a, a^\dagger$ and $b, b^\dagger$ [24]:

$$[a, a^\dagger] = [b, b^\dagger] = 1\!\!1 \ . \tag{5.1}$$

All other commutators involving these operators are zero.

The Hamiltonian describing the Landau levels is

$$\begin{aligned} H &= \hbar\omega(a^\dagger a + 1/2) \ , & (5.2) \\ \omega &= \frac{eB}{mc} = \text{cyclotron frequency} \ . & (5.3) \end{aligned}$$

Thus H commutes with the b-oscillators.

The a- and b- oscillators separately describe a Groenewold-Moyal plane since for example

$$\left[\frac{b + b^\dagger}{\sqrt{2}}, \frac{b - b^\dagger}{i\sqrt{2}}\right] = i1\!\!1 \ . \tag{5.4}$$

We can hence identify $(b + b^\dagger)/\sqrt{2}$ with $\hat{x}_1/l$, $(b - b^\dagger)/(i\sqrt{2})$ with $\hat{x}_2/l$ and $\theta_{\mu\nu}$ with $l^2\epsilon_{\mu\nu}$ ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \epsilon_{01} = +1$) where the scale factor l is the magnetic length:

$$l = \frac{1}{\sqrt{|e|B}} \ . \tag{5.5}$$

The a-oscillators give the discrete energy levels of the charged particle while the b-oscillators are associated with the coordinates of the plane $\mathbb{R}^2$. In fact, when only the lowest Landau levels are excited, it can be readily proved that $\hat{x}_\mu$ are the projections of the exact spatial coordinates to the subspace spanned by these levels. They become commuting coordinates when $B \to \infty$. In that limit, $\omega \to \infty$ so that the approximation of spatial coordinates by $\hat{x}_\mu$ becomes exact. The operators $\hat{x}_\mu$ are called the "guiding centre" coordinates.

When just the lowest Landau level is excited, the Hilbert space is

$$\mathcal{H}_1 \otimes \mathcal{H}_\infty \ , \tag{5.6}$$

where $\mathcal{H}_1$ has the vacuum state $|0\rangle$ of the a-oscillator as basis,

$$a|0\rangle = 0 \ , \tag{5.7}$$

and $\mathcal{H}_\infty$ is the Fock space of the b-oscillators. The observables are described by the algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ ($\theta_{\mu\nu} = l^2\epsilon_{\mu\nu}$) generated by $\hat{x}_\mu$.

When N Landau levels are excited, $\mathcal{H}_1$ becomes the $(N+1)$-dimensional Hilbert space $\mathcal{H}_{N+1}$ with basis

$$|0\rangle\ ,\ \frac{(a^\dagger)^k}{\sqrt{k!}}|0\rangle, \quad k=1,...,N \ . \tag{5.8}$$

The $(N+1)\times(N+1)$ matrix algebra Mat_{N+1} acts on $\mathcal{H}_{N+1}$:

$$Mat_{N+1}\mathcal{H}_{N+1} \subseteq \mathcal{H}_{N+1}\ . \tag{5.9}$$

The full Hilbert space is

$$\mathcal{H}_{N+1}\otimes\mathcal{H}_\infty\ . \tag{5.10}$$

The observables are thus described by the noncommutative algebra $Mat_{N+1}\otimes\mathcal{A}_\theta(\mathbb{R}^2)$.

The algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ admits the action of the diffeomorphism group $\mathcal{D}$ provided the coproduct for the latter is deformed. Although the quantum Hall system is non-relativistic, we can perhaps impose the dogma that the underlying spacetime algebra preserves its automorphism group in the process of deformation. If we do so, the statistics of the excitations described by (5.8) are also deformed.

We argue elsewhere [20] that at the second-quantized level, such excitations do not show UV-IR mixing. That is another good reason for the adoption of the deformed coproduct and statistics.

But the physical implications of this approach remain to be explored.

6 Remarks on Phenomenology

The most striking effects appear to be associated with violations of Pauli principle, and they can be subjected to stringent experimental tests. For example, life times for Pauli forbidden processes like $^{16}O \rightarrow ^{16}\tilde{O}$ or $^{12}C \rightarrow ^{12}\tilde{C}$, where $^{16}\tilde{O}$ ($^{12}\tilde{C}$) are nuclear configurations with an extra nucleon in the (filled) $1S_{1/2}$ shell, are presently found to be longer than 10^{27} years (90 % C.L.) [16,17]. Here we indicate how such transitions can arise by studying a very simple example: that of a free (twisted) fermion field. Spin effects are ignored as they are not important in this context.

So let $a(p)$ and $a^\dagger(p)$ be twisted fermionic creation and annihilation operators for momentum p. They can be written in the form (4.1) and (4.4) where $c(p)$ and its adjoint are fermionic oscillators for $\theta^{\mu\nu}=0$. A (twisted) single particle wave packet state $|\alpha\rangle$ is created from the vacuum by the operator

$$\langle a^\dagger,\alpha\rangle = \int \frac{d^dp}{2p_0}\alpha(p)a^\dagger(p)\ . \tag{6.1}$$

Thus

$$|\alpha\rangle = \langle a^\dagger, \alpha\rangle|0\rangle \tag{6.2}$$
$$= \langle c^\dagger, \alpha\rangle|0\rangle \ , \tag{6.3}$$
$$\langle c^\dagger, \alpha\rangle = \int \frac{d^d p}{2p_0}\alpha(p)c^\dagger(p) \ . \tag{6.4}$$

Hence with

$$\int \frac{d^d p}{2p_0}|\alpha(p)|^2 = 1 \ , \tag{6.5}$$

$|\alpha\rangle$ is normalized to unity:

$$\langle\alpha|\alpha\rangle = 1 \ . \tag{6.6}$$

We can approximate a vector with sharp momentum $\vec{p}$ with arbitrary precision with a function α peaked at $\vec{p}$ and normalized to 1. A Gaussian α is sufficient for this purpose.

Consider next the two-particle state vector

$$|\alpha,\alpha\rangle = \langle a^\dagger, \alpha\rangle\langle a^\dagger, \alpha\rangle|0\rangle \tag{6.7}$$
$$= \int \frac{d^d p_1}{2p_{10}}\frac{d^d p_2}{2p_{20}} e^{-\frac{i}{2}p_{1\mu}\theta^{\mu\nu}p_{2\nu}}\alpha(p_1)\alpha(p_2)c^\dagger(p_1)c^\dagger(p_2)|0\rangle \ . \tag{6.8}$$

This vector is identically zero if $\theta^{\mu\nu} = 0$ as required by Pauli principle.

But this vector is not zero if $\theta^{\mu\nu} \neq 0$, as shown for example by its non-vanishing norm $N(\alpha,\alpha)$:

$$N^2(\alpha,\alpha) = \langle\alpha,\alpha|\alpha,\alpha\rangle \tag{6.9}$$
$$= \int \frac{d^d p_1}{2p_{10}}\frac{d^d p_2}{2p_{20}}(\bar{\alpha}(p_1)\alpha(p_1))(\bar{\alpha}(p_2)\alpha(p_2))(1 - e^{-ip_{1\mu}\theta^{\mu\nu}p_{2\nu}}) \ . \tag{6.10}$$

$N^2(\alpha,\alpha) \neq 0$ for $\alpha \neq 0$ as can be seen from the following argument. We have

$$\int \frac{d^d p_1}{2p_{10}}\frac{d^d p_2}{2p_{20}}(\bar{\alpha}(p_1)\alpha(p_1))(\bar{\alpha}(p_2)\alpha(p_2))\sin(p_{1\mu}\theta^{\mu\nu}p_{2\nu}) = 0 \tag{6.11}$$

since the integrand is odd under the interchange of $p_1 \leftrightarrow p_2$. Hence

$$N^2(\alpha,\alpha) = \int \frac{d^d p_1}{2p_{10}}\frac{d^d p_2}{2p_{20}}(\bar{\alpha}(p_1)\alpha(p_1))(\bar{\alpha}(p_2)\alpha(p_2))[1 - \cos(p_{1\mu}\theta^{\mu\nu}p_{2\nu})] \ . \tag{6.12}$$

This is strictly positive for $\alpha \neq 0$ since $1-\cos(p_{1\mu}\theta^{\mu\nu}p_{2\nu}) \geq 0$ for $\theta^{\mu\nu} \neq 0$ and vanishes only on a zero-measure set of p_1, p_2. Note from (6.12) that $N(\alpha,\alpha)$ is $O(\theta)$.

We can normalize $|\alpha,\alpha\rangle$:

$$|\alpha,\alpha) = |\alpha,\alpha\rangle\frac{1}{N(\alpha,\alpha)} \ , \tag{6.13}$$
$$(\alpha,\alpha|\alpha,\alpha) = 1 \ . \tag{6.14}$$

This vector, being of unit norm, remains in the Hilbert space even if $\theta^{\mu\nu} \to 0$. But the scalar product of $|\alpha,\alpha)$ with the fermionic Fock space state $c^\dagger(p_1)c^\dagger(p_2)|0\rangle$ is undefined in the limit $\theta^{\mu\nu} \to 0$. Thus

$$\langle 0|c(p_2)c(p_1)|\alpha,\alpha) = -2i\alpha(p_1)\alpha(p_2)\frac{\sin(p_{1\mu}\theta^{\mu\nu}p_{2\nu}/2)}{N(\alpha,\alpha)} \ . \tag{6.15}$$

Since $N(\alpha,\alpha)$ is $O(\theta)$, the limit of this expression as $\theta^{\mu\nu} \to 0$ depends on the manner in which $\theta^{\mu\nu}$ goes to zero. This means that $|\alpha,\alpha)$ has different expansions in the Fock space basis depending on the way in which $\theta^{\mu\nu}$ becomes zero, that is it approaches different standard fermionic vectors in the Hilbert space depending on this limit. We do not know how to interpret this result.

Generalizing, we have the vectors

$$|\underbrace{\alpha,\alpha,...,\alpha}_{N \text{ factors}}\rangle = \langle a^\dagger,\alpha\rangle^N|0\rangle \ , \tag{6.16}$$

which after normalization become $|\alpha,\alpha,\ldots,\alpha)$, $(\alpha,\ldots,\alpha|\alpha,\ldots,\alpha) = 1$. These vectors span a Hilbert space $\mathcal{H}_S$ of symmetric vectors when $\theta^{\mu\nu} \to 0$.

Now consider for example

$$|\beta,\gamma\rangle = \langle a^\dagger,\beta\rangle\langle a^\dagger,\gamma\rangle|0\rangle, \quad \beta \neq \gamma. \tag{6.17}$$

We have

$$\langle\beta,\gamma|\alpha,\alpha) = \int \frac{d^dp_1}{2p_{10}}\frac{d^dp_2}{2p_{20}}(\bar\beta(p_1)\alpha(p_1))(\bar\gamma(p_2)\alpha(p_2))[1-e^{-ip_{1\mu}\theta^{\mu\nu}p_{2\nu}}]\frac{1}{N(\alpha,\alpha)} \ . \tag{6.18}$$

This overlap amplitude is not in general zero. Thus transitions are possible between Pauli-principle allowed state vectors $|\beta,\gamma\rangle$ and Pauli-principle forbidden state vectors $|\alpha,\alpha)$.

It is important to note that the mean energy and momentum in these new states are nothing outrageous. In fact, as one can see from (6.8), the mean value of P_μ in $|\alpha,\alpha,\cdots,\alpha)$ can be made arbitrarily close to Np_μ by choosing a Gaussian for α which is suitably peaked at p_μ.

In conventional Fock space, by Pauli principle, there is no fermionic state vector with energy-momentum Np_μ $(N \geq 2)$. This shows rather clearly that Pauli principle is violated when $\theta^{\mu\nu} \neq 0$.

We plan to further discuss the theory and phenomenology of these exotic states and associated transitions elsewhere.

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